6.3.1 The linearity property

As for Laplace transforms, a fundamental property of the z transform is its linearity, which may be stated as follows.

If $\{x_k\}$ and $\{y_k\}$ are sequences having z transforms $X(z)$ and $Y(z)$ respectively and if α and β are any constants, real or complex, then

$$\mathscr{Z}\{\alpha x_k + \beta y_k\} = \alpha\mathscr{Z}\{x_k\} + \beta\mathscr{Z}\{y_k\} = \alpha X(z) + \beta Y(z) \tag{6.12}$$

As a consequence of this property, we say that the z-transform operator $\mathscr{Z}$ is a **linear operator**. A proof of the property follows readily from the definition (6.4), since

$$\mathscr{Z}\{\alpha x_k + \beta y_k\} = \sum_{k=0}^{\infty} \frac{\alpha x_k + \beta y_k}{z^k} = \alpha \sum_{k=0}^{\infty} \frac{x_k}{z^k} + \beta \sum_{k=0}^{\infty} \frac{y_k}{z^k}$$
$$= \alpha X(z) + \beta Y(z)$$

The region of existence of the z transform, in the z plane, of the linear sum will be the intersection of the regions of existence (that is, the region common to both) of the individual z transforms $X(z)$ and $Y(z)$.

Example 6.5

The continuous-time function $f(t) = \cos \omega t\, H(t)$, ω a constant, is sampled in the idealized sense at intervals T to generate the sequence $\{\cos k\omega T\}$. Determine the z transform of the sequence.

Solution

Using the result $\cos k\omega T = \frac{1}{2}(e^{jk\omega T} + e^{-jk\omega T})$ and the linearity property, we have

$$\mathscr{Z}\{\cos k\omega T\} = \mathscr{Z}\{\tfrac{1}{2}e^{jk\omega T} + \tfrac{1}{2}e^{-jk\omega T}\} = \tfrac{1}{2}\mathscr{Z}\{e^{jk\omega T}\} + \tfrac{1}{2}\mathscr{Z}\{e^{-jk\omega T}\}$$

Using (6.7) and noting that $|e^{jk\omega T}| = |e^{-jk\omega T}| = 1$ gives

$$\mathscr{Z}\{\cos k\omega T\} = \frac{1}{2}\frac{z}{z - e^{j\omega T}} + \frac{1}{2}\frac{z}{z - e^{-j\omega T}} \quad (|z| > 1)$$
$$= \frac{1}{2}\frac{z(z - e^{-j\omega T}) + z(z - e^{j\omega T})}{z^2 - (e^{j\omega T} + e^{-j\omega T})z + 1}$$

leading to the z-transform pair

$$\mathscr{Z}\{\cos k\omega T\} = \frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1} \quad (|z| > 1) \tag{6.13}$$

In a similar manner to Example 6.5, we can verify the z-transform pair

$$\mathscr{Z}\{\sin k\omega T\} = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1} \quad (|z| > 1) \tag{6.14}$$

and this is left as an exercise for the reader (see Exercise 3).

Check that in MATLAB the commands

```
syms k z ω T
ztrans(cos(k*ω*T));
pretty(simple(ans))
```

return the transform given in (6.13) and that the MAPLE commands:

```
ztrans(cos(k*ω*T),k,z);
simplify(%);
```

do likewise.

6.3.2 The first shift property (delaying)

In this and the next section we introduce two properties relating the z transform of a sequence to the z transform of a shifted version of the same sequence. In this section we consider a delayed version of the sequence $\{x_k\}$, denoted by $\{y_k\}$, with

$$y_k = x_{k-k_0}$$

Here k_0 is the number of steps in the delay; for example, if $k_0 = 2$ then $y_k = x_{k-2}$, so that

$$y_0 = x_{-2}, \quad y_1 = x_{-1}, \quad y_2 = x_0, \quad y_3 = x_1$$

and so on. Thus the sequence $\{y_k\}$ is simply the sequence $\{x_k\}$ moved backward, or delayed, by two steps. From the definition (6.1),

$$\mathscr{Z}\{y_k\} = \sum_{k=0}^{\infty} \frac{y_k}{z^k} = \sum_{k=0}^{\infty} \frac{x_{k-k_0}}{z^k} = \sum_{p=-k_0}^{\infty} \frac{x_p}{z^{p+k_0}}$$

where we have written $p = k - k_0$. If $\{x_k\}$ is a causal sequence, so that $x_p = 0$ $(p < 0)$, then

$$\mathscr{Z}\{y_k\} = \sum_{p=0}^{\infty} \frac{x_p}{z^{p+k_0}} = \frac{1}{z^{k_0}} \sum_{p=0}^{\infty} \frac{x_p}{z^p} = \frac{1}{z^{k_0}} X(z)$$

where $X(z)$ is the z transform of $\{x_k\}$.

We therefore have the result

$$\mathscr{Z}\{x_{k-k_0}\} = \frac{1}{z^{k_0}} \mathscr{Z}\{x_k\} \tag{6.15}$$

which is referred to as the **first shift property** of z transforms.

If $\{x_k\}$ represents the sampled form, with uniform sampling interval T, of the continuous signal $x(t)$ then $\{x_{k-k_0}\}$ represents the sampled form of the continuous signal $x(t - k_0T)$ which, as illustrated in Figure 6.2, is the signal $x(t)$ delayed by a multiple k_0 of the sampling interval T. The reader will find it of interest to compare this result with the results for the Laplace transforms of integrals (5.16).

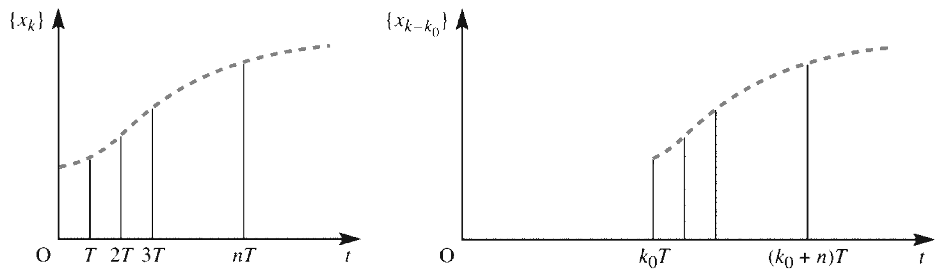

Figure 6.2 Sequence and its shifted form.

Example 6.6

The causal sequence $\{x_k\}$ is generated by

$$x_k = (\tfrac{1}{2})^k \quad (k \geqslant 0)$$

Determine the z transform of the shifted sequence $\{x_{k-2}\}$.

Solution By the first shift property,

$$\mathscr{Z}\{x_{k-2}\} = \frac{1}{z^2}\mathscr{Z}\{(\tfrac{1}{2})^k\}$$

which, on using (6.4), gives

$$\mathscr{Z}\{x_{k-2}\} = \frac{1}{z^2}\frac{z}{z-\frac{1}{2}} \quad (|z| > \tfrac{1}{2}) \quad = \frac{1}{z^2}\frac{2z}{2z-1} = \frac{2}{z(2z-1)} \quad (|z| > \tfrac{1}{2})$$

We can confirm this result by direct use of the definition (6.1). From this, and the fact that $\{x_k\}$ is a causal sequence,

$$\{x_{k-2}\} = \{x_{-2}, x_{-1}, x_0, x_1, \ldots\} = \{0, 0, 1, \tfrac{1}{2}, \tfrac{1}{4}, \ldots\}$$

Thus,

$$\mathscr{Z}\{x_{k-2}\} = 0 + 0 + \frac{1}{z^2} + \frac{1}{2z^3} + \frac{1}{4z^4} + \ldots = \frac{1}{z^2}\left(1 + \frac{1}{2z} + \frac{1}{4z^2} + \ldots\right)$$

$$= \frac{1}{z^2}\frac{z}{z-\frac{1}{2}} \quad (|z| > \tfrac{1}{2}) \quad = \frac{z}{z(2z-1)} \quad (|z| > \tfrac{1}{2})$$

6.3.3 The second shift property (advancing)

In this section we seek a relationship between the z transform of an advanced version of a sequence and that of the original sequence. First we consider a single-step advance. If $\{y_k\}$ is the single-step advanced version of the sequence $\{x_k\}$ then $\{y_k\}$ is generated by

$$y_k = x_{k+1} \quad (k \geqslant 0)$$

Then

$$\mathscr{Z}\{y_k\} = \sum_{k=0}^{\infty} \frac{y_k}{z^k} = \sum_{k=0}^{\infty} \frac{x_{k+1}}{z^k} = z\sum_{k=0}^{\infty} \frac{x_{k+1}}{z^{k+1}}$$

and putting $p = k + 1$ gives

$$\mathscr{Z}\{y_k\} = z\sum_{p=1}^{\infty} \frac{x_p}{z^p} = z\left(\sum_{p=0}^{\infty} \frac{x_p}{z^p} - x_0\right) = zX(z) - zx_0$$

where $X(z)$ is the z transform of $\{x_k\}$.

We therefore have the result

$$\mathscr{Z}\{x_{k+1}\} = zX(z) - zx_0 \qquad \textbf{(6.16)}$$

In a similar manner it is readily shown that for a two-step advanced sequence $\{x_{k+2}\}$

$$\mathscr{Z}\{x_{k+2}\} = z^2X(z) - z^2x_0 - zx_1 \qquad \textbf{(6.17)}$$

Note the similarity in structure between (6.16) and (6.17) on the one hand and those for the Laplace transforms of first and second derivatives (Section 5.3.1). In general, it is readily proved by induction that for a k_0-step advanced sequence $\{x_{k+k_0}\}$

$$\mathscr{Z}\{x_{k+k_0}\} = z^{k_0}X(z) - \sum_{n=0}^{k_0-1} x_n z^{k_0-n} \qquad \textbf{(6.18)}$$

In Section 6.5.2 we shall use these results to solve difference equations.

6.3.4 Some further properties

In this section we shall state some further useful properties of the z transform, leaving their verification to the reader as Exercises 9 and 10.

(i) *Multiplication by a^k*

If $Z\{x_k\} = X(z)$ then for a constant a

$$\mathscr{Z}\{a^k x_k\} = X(a^{-1}z) \qquad \textbf{(6.19)}$$

(ii) *Multiplication by k^n*

If $\mathscr{Z}\{x_k\} = X(z)$ then for a positive integer n

$$\mathscr{Z}\{k^n x_k\} = \left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)^n X(z) \qquad \textbf{(6.20)}$$

Note that in (6.20) the operator $-z\,\mathrm{d}/\mathrm{d}z$ means 'first differentiate with respect to z and then multiply by $-z$'. Raising to the power of n means 'repeat the operation n times'.

(iii) *Initial-value theorem*

If $\{x_k\}$ is a sequence with z transform $X(z)$ then the initial-value theorem states that

$$\lim_{z\to\infty} X(z) = x_0 \tag{6.21}$$

(iv) *Final-value theorem*

If $\{x_k\}$ is a sequence with z transform $X(z)$ then the final-value theorem states that

$$\lim_{k\to\infty} x_k = \lim_{z\to 1}(1 - z^{-1})X(z) \tag{6.22}$$

provided that the poles of $(1 - z^{-1})X(z)$ are inside the unit circle.

6.3.5 Table of *z* transforms

It is appropriate at this stage to draw together the results proved so far for easy access. This is done in the form of a table in Figure 6.3.

Figure 6.3 A short table of z transforms.

$\{x_k\}$ $(k \geqslant 0)$	$\mathcal{Z}\{x_k\}$	Region of existence
$x_k = \begin{cases} 1 & (k = 0) \\ 0 & (k > 0) \end{cases}$ (unit pulse sequence)	1	All z
$x_k = 1$ (unit step sequence)	$\dfrac{z}{z-1}$	$\lvert z\rvert > 1$
$x_k = a^k$ (a constant)	$\dfrac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$x_k = k$	$\dfrac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$x_k = ka^{k-1}$ (a constant)	$\dfrac{z}{(z-a)^2}$	$\lvert z\rvert > a$
$x_k = \mathrm{e}^{-kT}$ (T constant)	$\dfrac{z}{z-\mathrm{e}^{-T}}$	$\lvert z\rvert > \mathrm{e}^{-T}$
$x_k = \cos k\omega T$ (ω, T constants)	$\dfrac{z(z-\cos\omega T)}{z^2 - 2z\cos\omega T + 1}$	$\lvert z\rvert > 1$
$x_k = \sin k\omega T$ (ω, T constants)	$\dfrac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}$	$\lvert z\rvert > 1$

6.3.6 Exercises

Check your answers using MATLAB or MAPLE whenever possible.

3 Use the method of Example 6.5 to confirm (6.14), namely

$$\mathscr{Z}\{\sin k\omega T\} = \frac{z\sin \omega T}{z^2 - 2z\cos \omega T + 1}$$

where ω and T are constants.

4 Use the first shift property to calculate the z transform of the sequence $\{y_k\}$, with

$$y_k = \begin{cases} 0 & (k < 3) \\ x_{k-3} & (k \geqslant 3) \end{cases}$$

where $\{x_k\}$ is causal and $x_k = (\frac{1}{2})^k$. Confirm your result by direct evaluation of $\mathscr{Z}\{y_k\}$ using the definition of the z transform.

5 Determine the z transforms of the sequences

(a) $\{(-\frac{1}{5})^k\}$ (b) $\{\cos k\pi\}$

6 Determine $\mathscr{Z}\{(\frac{1}{2})^k\}$. Using (6.6), obtain the z transform of the sequence $\{k(\frac{1}{2})^k\}$.

7 Show that for a constant α

(a) $\mathscr{Z}\{\sinh k\alpha\} = \dfrac{z\sinh\alpha}{z^2 - 2z\cosh\alpha + 1}$

(b) $\mathscr{Z}\{\cosh k\alpha\} = \dfrac{z^2 - z\cosh\alpha}{z^2 - 2z\cosh\alpha + 1}$

8 Sequences are generated by sampling a causal continuous-time signal $u(t)$ $(t \geqslant 0)$ at uniform intervals T. Write down an expression for u_k, the general term of the sequence, and calculate the corresponding z transform when $u(t)$ is

(a) e^{-4t} (b) $\sin t$ (c) $\cos 2t$

9 Prove the initial- and final-value theorems given in (6.21) and (6.22).

10 Prove the multiplication properties given in (6.19) and (6.20).

6.4 The inverse *z* transform

In this section we consider the problem of recovering a causal sequence $\{x_k\}$ from knowledge of its z transform $X(z)$. As we shall see, the work on the inversion of Laplace transforms in Section 5.2.7 will prove a valuable asset for this task.

Formally the symbol $\mathscr{Z}^{-1}[X(z)]$ denotes a causal sequence $\{x_k\}$ whose z transform is $X(z)$; that is,

$$\text{if } \mathscr{Z}\{x_k\} = X(z) \quad \text{then} \quad \{x_k\} = \mathscr{Z}^{-1}[X(z)]$$

This correspondence between $X(z)$ and $\{x_k\}$ is called the **inverse *z* transformation**, $\{x_k\}$ being the **inverse transform** of $X(z)$, and $\mathscr{Z}^{-1}$ being referred to as the **inverse *z*-transform operator**.

As for the Laplace transforms in Section 5.2.8, the most obvious way of finding the inverse transform of $X(z)$ is to make use of a table of transforms such as that given in Figure 6.3. Sometimes it is possible to write down the inverse transform directly from the table, but more often than not it is first necessary to carry out some algebraic manipulation on $X(z)$. In particular, we frequently need to determine the inverse transform of a rational expression of the form $P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials in z. In such cases the procedure, as for Laplace transforms, is first to resolve the expression, or a revised form of the expression, into partial fractions and then to use the table of transforms. We shall now illustrate the approach through some examples.