6.4.1 Inverse techniques

Example 6.7 Find

$$\mathscr{Z}^{-1}\left[\frac{z}{z-2}\right]$$

Solution From Figure 6.3, we see that $z/(z-2)$ is a special case of the transform $z/(z-a)$, with $a = 2$. Thus

$$\mathscr{Z}^{-1}\left[\frac{z}{z-2}\right] = \{2^k\}$$

Example 6.8 Find

$$\mathscr{Z}^{-1}\left[\frac{z}{(z-1)(z-2)}\right]$$

Solution Guided by our work on Laplace transforms, we might attempt to resolve

$$Y(z) = \frac{z}{(z-1)(z-2)}$$

into partial fractions. This approach does produce the correct result, as we shall show later. However, we notice that most of the entries in Figure 6.3 contain a factor z in the numerator of the transform. We therefore resolve

$$\frac{Y(z)}{z} = \frac{1}{(z-1)(z-2)}$$

into partial fractions, as

$$\frac{Y(z)}{z} = \frac{1}{z-2} - \frac{1}{z-1}$$

so that

$$Y(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

Then using the result $\mathscr{Z}^{-1}[z/(z-a)] = \{a^k\}$ together with the linearity property, we have

$$\mathscr{Z}^{-1}[Y(z)] = \mathscr{Z}^{-1}\left(\frac{z}{z-2} - \frac{z}{z-1}\right) = \mathscr{Z}^{-1}\left(\frac{z}{z-2}\right) - \mathscr{Z}^{-1}\left(\frac{z}{z-1}\right)$$

$$= \{2^k\} - \{1^k\} \quad (k \geqslant 0)$$

so that

$$\mathscr{Z}^{-1}\left[\frac{z}{(z-1)(z-2)}\right] = \{2^k - 1\} \quad (k \geqslant 0) \qquad \textbf{(6.23)}$$

Suppose that in Example 6.8 we had not thought so far ahead and we had simply resolved $Y(z)$, rather than $Y(z)/z$, into partial fractions. Would the result be the same? The answer of course is 'yes', as we shall now show. Resolving

$$Y(z) = \frac{z}{(z-1)(z-2)}$$

into partial fractions gives

$$Y(z) = \frac{2}{z-2} - \frac{1}{z-1}$$

which may be written as

$$Y(z) = \frac{1}{z}\frac{2z}{z-2} - \frac{1}{z}\frac{z}{z-1}$$

Since

$$\mathscr{Z}^{-1}\left[\frac{2z}{z-2}\right] = 2\mathscr{Z}^{-1}\left(\frac{z}{z-2}\right) = 2\{2^k\}$$

it follows from the first shift property (6.15) that

$$\mathscr{Z}^{-1}\left[\frac{1}{z}\frac{2z}{z-2}\right] = \begin{cases} \{2 \cdot 2^{k-1}\} & (k > 0) \\ 0 & (k = 0) \end{cases}$$

Similarly,

$$\mathscr{Z}^{-1}\left[\frac{1}{z}\frac{z}{z-1}\right] = \begin{cases} \{1^{k-1}\} = \{1\} & (k > 0) \\ 0 & (k = 0) \end{cases}$$

Combining these last two results, we have

$$\mathscr{Z}^{-1}[Y(z)] = \mathscr{Z}^{-1}\left[\frac{1}{z}\frac{2z}{z-2}\right] - \mathscr{Z}^{-1}\left[\frac{1}{z}\frac{z}{z-1}\right]$$

$$= \begin{cases} \{2^k - 1\} & (k > 0) \\ 0 & (k = 0) \end{cases}$$

which, as expected, is in agreement with the answer obtained in Example 6.8.

We can see that adopting this latter approach, while producing the correct result, involved extra effort in the use of a shift theorem. When possible, we avoid this by 'extracting' the factor z as in Example 6.8, but of course this is not always possible, and recourse may be made to the shift property, as Example 6.9 illustrates.

The inverse z-transform $\{x_k\}$ of $X(z)$ is returned in MATLAB using the command

```
iztrans(X(z),k)
```

[*Note*: The command `iztrans(X(z))` by itself returns the inverse transform expressed in terms of n rather than k.]

For the z-transform in Example 6.8 the MATLAB command

```
iztrans(z/((z-1)*(z-2)),k)
```

returns

```
ans=-1+2^k
```

as required.

The inverse z-transform can be performed in MAPLE using the `invztrans` function, so that the command

```
invztrans(z/(z^2-3*z+2)z,k);
```

also returns the answer

```
2^k - 1
```

Example 6.9 Find

$$\mathscr{Z}^{-1}\left[\frac{2z+1}{(z+1)(z-3)}\right]$$

Solution In this case there is no factor z available in the numerator, and so we must resolve

$$Y(z) = \frac{2z+1}{(z+1)(z-3)}$$

into partial fractions, giving

$$Y(z) = \tfrac{1}{4}\frac{1}{z+1} + \tfrac{7}{4}\frac{1}{z-3} = \tfrac{1}{4}\frac{1}{z}\frac{z}{z+1} + \tfrac{7}{4}\frac{1}{z}\frac{z}{z-3}$$

Since

$$\mathscr{Z}^{-1}\left[\frac{z}{z+1}\right] = \{(-1)^k\} \quad (k \geqslant 0)$$

$$\mathscr{Z}^{-1}\left[\frac{z}{z-3}\right] = \{3^k\} \quad (k \geqslant 0)$$

it follows from the first shift property (6.15) that

$$\mathscr{Z}^{-1}\left[\frac{1}{z}\frac{z}{z+1}\right] = \begin{cases} \{(-1)^{k-1}\} & (k > 0) \\ 0 & (k = 0) \end{cases}$$

$$\mathscr{Z}^{-1}\left[\frac{1}{z}\frac{z}{z-3}\right] = \begin{cases} 3^{k-1} & (k > 0) \\ 0 & (k = 0) \end{cases}$$

Then, from the linearity property,

$$\mathscr{Z}^{-1}[Y(z)] = \tfrac{1}{4}\mathscr{Z}^{-1}\left[\frac{1}{z}\frac{z}{z+1}\right] + \tfrac{7}{4}\mathscr{Z}^{-1}\left[\frac{1}{z}\frac{z}{z-3}\right]$$

giving

$$\mathscr{Z}^{-1}\left[\frac{2z+1}{(z+1)(z-3)}\right] = \begin{cases}\{\frac{1}{4}(-1)^{k-1}+\frac{7}{4}3^{k-1}\} & (k > 0)\\ 0 & (k = 0)\end{cases}$$

In MATLAB the command

```
iztrans((2*z+1)/((z+1)*(z-3)),k)
```

returns

```
ans=-1/3*charfcn[0](k)-1/4*(-1)^k+7/12*3^k
```

[*Note*: The charfcn function is the characteristic function of the **set** *A*, and is defined to be

$$\text{charfcn}[A](k) = \begin{cases}1 & \text{if } k \text{ is in } A\\ 0 & \text{if } k \text{ is not in } A\end{cases}$$

Thus charfcn $[0](k) = 1$ if $k = 0$ and 0 otherwise.]

It is left as an exercise to confirm that the answer provided using MATLAB concurs with the calculated answer.

It is often the case that the rational function $P(z)/Q(z)$ to be inverted has a quadratic term in the denominator. Unfortunately, in this case there is nothing resembling the first shift theorem of the Laplace transform which, as we saw in Section 5.2.9, proved so useful in similar circumstances. Looking at Figure 6.3, the only two transforms with quadratic terms in the denominator are those associated with the sequences $\{\cos k\omega T\}$ and $\{\sin k\omega T\}$. In practice these prove difficult to apply in the inverse form, and a 'first principles' approach is more appropriate. We illustrate this with two examples, demonstrating that all that is really required is the ability to handle complex numbers at the stage of resolution into partial fractions.

Example 6.10

Invert the *z* transform

$$Y(z) = \frac{z}{z^2 + a^2}$$

where *a* is a real constant.

Solution

In view of the factor *z* in the numerator, we resolve $Y(z)/z$ into partial fractions, giving

$$\frac{Y(z)}{z} = \frac{1}{z^2 + a^2} = \frac{1}{(z+\mathrm{j}a)(z-\mathrm{j}a)} = \frac{1}{\mathrm{j}2a}\frac{1}{(z-\mathrm{j}a)} - \frac{1}{\mathrm{j}2a}\frac{1}{(z+\mathrm{j}a)}$$

That is

$$Y(z) = \frac{1}{\mathrm{j}2a}\left(\frac{z}{z-\mathrm{j}a} - \frac{z}{z+\mathrm{j}a}\right)$$

Using the result $\mathscr{Z}^{-1}[z/(z-a)] = \{a^k\}$, we have

$$\mathscr{Z}^{-1}\left[\frac{z}{z-\mathrm{j}a}\right] = \{(\mathrm{j}a)^k\} = \{\mathrm{j}^k a^k\}$$

$$\mathscr{Z}^{-1}\left[\frac{z}{z+\mathrm{j}a}\right] = \{(-\mathrm{j}a)^k\} = \{(-\mathrm{j})^k a^k\}$$

From the relation $\mathrm{e}^{\mathrm{j}\theta} = \cos\theta + \mathrm{j}\sin\theta$, we have

$$\mathrm{j} = \mathrm{e}^{\mathrm{j}\pi/2}, \qquad -\mathrm{j} = \mathrm{e}^{-\mathrm{j}\pi/2}$$

so that

$$\mathscr{Z}^{-1}\left[\frac{z}{z-\mathrm{j}a}\right] = \{a^k(\mathrm{e}^{\mathrm{j}\pi/2})^k\} = \{a^k\mathrm{e}^{\mathrm{j}k\pi/2}\} = \{a^k(\cos\tfrac{1}{2}k\pi + \mathrm{j}\sin\tfrac{1}{2}k\pi)\}$$

$$\mathscr{Z}^{-1}\left[\frac{z}{z+\mathrm{j}a}\right] = \{a^k(\cos\tfrac{1}{2}k\pi - \mathrm{j}\sin\tfrac{1}{2}k\pi)\}$$

The linearity property then gives

$$\begin{aligned}\mathscr{Z}^{-1}[Y(z)] &= \left\{\frac{a^k}{\mathrm{j}2a}(\cos\tfrac{1}{2}k\pi + \mathrm{j}\sin\tfrac{1}{2}k\pi - \cos\tfrac{1}{2}k\pi + \mathrm{j}\sin\tfrac{1}{2}k\pi)\right\}\\ &= \{a^{k-1}\sin\tfrac{1}{2}k\pi\}\end{aligned}$$

Whilst MATLAB or MAPLE may be used to obtain the inverse z transform when complex partial fractions are involved, it is difficult to convert results into a simple form, the difficult step being that of expressing complex exponentials in terms of trigonometric functions.

Example 6.11

Invert

$$Y(z) = \frac{z}{z^2 - z + 1}$$

Solution The denominator of the transform may be factorized as

$$z^2 - z + 1 = \left(z - \tfrac{1}{2} - \mathrm{j}\frac{\sqrt{3}}{2}\right)\left(z - \tfrac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2}\right)$$

In exponential form we have $\frac{1}{2} \pm \mathrm{j}\frac{1}{2}\sqrt{3} = \mathrm{e}^{\pm\mathrm{j}\pi/3}$, so the denominator may be written as

$$z^2 - z + 1 = (z - \mathrm{e}^{\mathrm{j}\pi/3})(z - \mathrm{e}^{-\mathrm{j}\pi/3})$$

We then have

$$\frac{Y(z)}{z} = \frac{1}{(z - \mathrm{e}^{\mathrm{j}\pi/3})(z - \mathrm{e}^{-\mathrm{j}\pi/3})}$$

which can be resolved into partial fractions as

$$\frac{Y(z)}{z} = \frac{1}{\mathrm{e}^{\mathrm{j}\pi/3} - \mathrm{e}^{-\mathrm{j}\pi/3}}\,\frac{1}{z - \mathrm{e}^{\mathrm{j}\pi/3}} + \frac{1}{\mathrm{e}^{-\mathrm{j}\pi/3} - \mathrm{e}^{\mathrm{j}\pi/3}}\,\frac{1}{z - \mathrm{e}^{-\mathrm{j}\pi/3}}$$

Noting that $\sin\theta = (e^{j\theta} - e^{-j\theta})/j2$, this reduces to

$$\frac{Y(z)}{z} = \frac{1}{j2\sin\frac{1}{3}\pi}\frac{z}{z - e^{j\pi/3}} - \frac{1}{j2\sin\frac{1}{3}\pi}\frac{z}{z - e^{-j\pi/3}}$$

$$= \frac{1}{j\sqrt{3}}\frac{z}{z - e^{j\pi/3}} - \frac{1}{j\sqrt{3}}\frac{z}{z - e^{-j\pi/3}}$$

Using the result $\mathscr{Z}^{-1}[z/(z-a)] = \{a^k\}$, this gives

$$\mathscr{Z}^{-1}[Y(z)] = \frac{1}{j\sqrt{3}}(e^{jk\pi/3} - e^{-jk\pi/3}) = \{2\sqrt{\tfrac{1}{3}}\sin\tfrac{1}{3}k\pi\}$$

We conclude this section with two further examples, illustrating the inversion technique applied to frequently occurring transform types.

Example 6.12

Find the sequence whose z transform is

$$F(z) = \frac{z^3 + 2z^2 + 1}{z^3}$$

Solution

$F(z)$ is unlike any z transform treated so far in the examples. However, it is readily expanded in a power series in z^{-1} as

$$F(z) = 1 + \frac{2}{z} + \frac{1}{z^3}$$

Using (6.4), it is then apparent that

$$\mathscr{Z}^{-1}[F(z)] = \{f_k\} = \{1, 2, 0, 1, 0, 0, \ldots\}$$

The MATLAB command

```
iztrans((z^3+2*z^2+1)/z^3,k)
```

returns

```
charfcn[0](k)+2*charfcn[1](k)+charfcn[3](k)
```

which corresponds to the sequence

$\{1, 2, 0, 1, 0, 0, \ldots\}$

Example 6.13

Find $\mathscr{Z}^{-1}[G(z)]$ where

$$G(z) = \frac{z(1 - e^{-aT})}{(z-1)(z - e^{-aT})}$$

where a and T are positive constants.

Solution

Resolving into partial fractions,

$$\frac{G(z)}{z} = \frac{1}{z-1} - \frac{1}{z - e^{-aT}}$$

giving

$$G(z) = \frac{1}{z-1} - \frac{1}{z - e^{-aT}}$$

Using the result $\mathscr{Z}^{-1}[z/(z-a)] = \{a^k\}$, we have

$$\mathscr{Z}^{-1}[G(z)] = \{(1 - e^{-akT})\} \quad (k \geqslant 0)$$

In this particular example $G(z)$ is the z transform of a sequence derived by sampling the continuous-time signal

$$f(t) = 1 - e^{-at}$$

at intervals T.

The MATLAB commands

```
syms k z a T
iztrans((z*(1-exp(-a*T)))/((z-1)*(z-exp(-a*T))),k);
pretty(simple(ans))
```

return

```
ans=1-exp(-aT)^k
```

In MAPLE the command

```
invztrans((z*(1-exp(-aT)))/((z-1)*(z-exp(-aT))),z,k);
```

returns

$$-\left(\frac{1}{e^{aT}}\right)^k + 1$$

6.4.2 Exercises

Confirm your answers using MATLAB or MAPLE whenever possible.

11 Invert the following z transforms. Give the general term of the sequence in each case.

(a) $\frac{z}{z-1}$ (b) $\frac{z}{z+1}$ (c) $\frac{z}{z-\frac{1}{2}}$

(d) $\frac{z}{3z+1}$ (e) $\frac{z}{z-j}$ (f) $\frac{z}{z+j\sqrt{2}}$

(g) $\frac{1}{z-1}$ (h) $\frac{z+2}{z+1}$

12 By first resolving $Y(z)/z$ into partial fractions, find $\mathscr{Z}^{-1}[Y(z)]$ when $Y(z)$ is given by

(a) $\frac{z}{(z-1)(z+2)}$ (b) $\frac{z}{(2z+1)(z-3)}$

(c) $\frac{z^2}{(2z+1)(z-1)}$ (d) $\frac{2z}{2z^2+z-1}$

(e) $\frac{z}{z^2+1}$ [*Hint*: $z^2+1 = (z+j)(z-j)$]

(f) $\frac{z}{z^2 - 2\sqrt{3}z + 4}$ (g) $\frac{2z^2-7z}{(z-1)^2(z-3)}$

(h) $\frac{z^2}{(z-1)^2(z^2-z+1)}$

13 Find $\mathscr{Z}^{-1}[Y(z)]$ when $Y(z)$ is given by

(a) $\frac{1}{z} + \frac{2}{z^7}$ (b) $1 + \frac{3}{z^2} - \frac{2}{z^9}$

(c) $\frac{3z+z^2+5z^5}{z^5}$ (d) $\frac{1+z}{z^3} + \frac{3z}{3z+1}$

(e) $\frac{2z^3+6z^2+5z+1}{z^2(2z+1)}$ (f) $\frac{2z^2-7z+7}{(z-1)^2(z-2)}$

(g) $\frac{z-3}{z^2-3z+2}$