Chapter 1.
Matrix Analysis :

determinant $|\boldsymbol{A}| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij}M_{ij} = \sum_{j=1}^{n} a_{ij}A_{ij}$ M_{ij}: minor A_{ij}: cofactor

$$\text{adj}\,\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^{\mathrm{T}} \qquad \boldsymbol{A}^{-1} = (\text{adj}\,\boldsymbol{A})/|\boldsymbol{A}|$$

$c(\lambda)$: **characteristic polynomial** of $\boldsymbol{A}$. $c(\lambda) = |\lambda \boldsymbol{I} - \boldsymbol{A}| = 0$

Jordan form of $\boldsymbol{A}$. $\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M} = \boldsymbol{J} = [J_1 \quad J_2 \quad | \ldots \quad J_p]$

for Jordan canonical form: $(\boldsymbol{A} - \lambda_i \boldsymbol{I})e_i^* = e_i$

State-space representation:

$$a_n\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = u(t)$$

$$x_1(t) = y(t), \quad \dot{x}_1(t) = x_2(t) \quad , \ldots, \ \dot{x}_{n-1}(t) = x_n(t)$$

$$\dot{x}_n = -\frac{a_{n-1}}{a_n}x_n - \frac{a_{n-2}}{a_n}x_{n-1} - \ldots - \frac{a_1}{a_n}x_2 - \frac{a_0}{a_n}x_1 + \frac{1}{a_n}u(t)$$

Chapter 5.
Laplace Transform

$$\mathscr{L}\{f(t)\} = F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$	Region of convergence
c, c a constant	$\frac{c}{s}$	$\text{Re}(s) > 0$
t	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
t^n, n a positive integer	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
e^{kt}, k a constant	$\frac{1}{s-k}$	$\text{Re}(s) > \text{Re}(k)$
$\sin at$, a a real constant	$\frac{a}{s^2+a^2}$	$\text{Re}(s) > 0$
$\cos at$, a a real constant	$\frac{s}{s^2+a^2}$	$\text{Re}(s) > 0$
$\mathrm{e}^{-kt}\sin at$, k and a real constants	$\frac{a}{(s+k)^2+a^2}$	$\text{Re}(s) > -k$
$\mathrm{e}^{-kt}\cos at$, k and a real constants	$\frac{s+k}{(s+k)^2+a^2}$	$\text{Re}(s) > -k$

Properties of the Laplace transform

if $\mathscr{L}\{f(t)\} = F(s)$, $\text{Re}(s) > \sigma_1$ and $\mathscr{L}\{g(t)\} = G(s)$, $\text{Re}(s) > \sigma_2$

Linearity: $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$, $\text{Re}(s) > \max(\sigma_1, \sigma_2)$

First shift theorem: $\mathscr{L}\{\mathrm{e}^{at}f(t)\} = F(s-a)$, $\text{Re}(s) > \sigma_1 + \text{Re}(a)$

Derivative of transform: $\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n}$, $(n = 1, 2, \ldots)$, $\text{Re}(s) > \sigma_1$

The inverse transform $\mathscr{L}^{-1}\{F(s)\} = f(t)H(t)$

Inversion using the first shift theorem $\mathscr{L}^{-1}\{F(s-a)\} = \mathrm{e}^{at}f(t)$

Convolution $f * g(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau$

$\mathscr{L}\{f * g(t)\} = F(s)G(s)$ $\mathscr{L}^{-1}\{F(s)G(s)\} = f * g(t)$

$\boldsymbol{M} = [e_1 \quad e_2 \quad e_3 \quad \ldots \quad e_n]$ **for canonical form:** $(\boldsymbol{A} - \lambda_i \boldsymbol{I})e_i = 0$

$\boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M} = \boldsymbol{\Lambda}$ $\boldsymbol{A} = \boldsymbol{M}\boldsymbol{\Lambda}\boldsymbol{M}^{-1}$

$\boldsymbol{\Lambda}$.canonical form $\boldsymbol{M}$.modal matrix

quadratic form V: $\boldsymbol{\Lambda} = \hat{\boldsymbol{M}}^{\mathrm{T}}\boldsymbol{A}\hat{\boldsymbol{M}}$ $\boldsymbol{x} = \hat{\boldsymbol{M}}\boldsymbol{y}$

$$\boldsymbol{x} = [x_1 \quad x_2 \quad \ldots \quad x_n]^{\mathrm{T}} \qquad \boldsymbol{y} = [y_1 \quad y_2 \quad \ldots \quad y_n]^{\mathrm{T}}$$

$$V = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}^{\mathrm{T}}\hat{\boldsymbol{M}}^{\mathrm{T}}\boldsymbol{A}\hat{\boldsymbol{M}}\boldsymbol{y} = \boldsymbol{y}^{\mathrm{T}}\boldsymbol{\Lambda}\boldsymbol{y}$$

single-input–single-output (SISO) system

$$a_n\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = u(t)$$

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u \qquad y = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$$

Taylor series

$$f(x+a) = f(a) + \frac{x}{1!}f^{(1)}(a) + \frac{x^2}{2!}f^{(2)}(a) + \ldots = \sum_{n=0}^{\infty}\frac{x^n}{n!}f^{(n)}(a)$$

power series about z_0

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots + a_r(z-z_0)^r + \ldots$$

Transforms of derivatives

$$\mathscr{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = sF(s) - f(0) \qquad \mathscr{L}\left\{\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right\} = s^2F(s) - sf(0) - f^{(1)}(0)$$

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f^{(1)}(0) - \ldots - f^{(n-1)}(0)$$

Transforms of integrals

$$\mathscr{L}\left\{\int_0^t f(\tau)\,\mathrm{d}\tau\right\} = \frac{1}{s}\mathscr{L}\{f(t)\} = \frac{1}{s}F(s)$$

Laplace transform of unit step function

$$\mathscr{L}\{H(t-a)\} = \frac{\mathrm{e}^{-as}}{s} \quad (a \geqslant 0) \qquad \mathscr{L}\{H(t)\} = \frac{1}{s}$$

The second shift theorem

If $\mathscr{L}\{f(t)\} = F(s)$ then $\mathscr{L}\{f(t-a)H(t-a)\} = \mathrm{e}^{-as}F(s)$

Inversion using the second shift theorem

$$\mathscr{L}^{-1}\{\mathrm{e}^{-as}F(s)\} = f(t-a)H(t-a)$$

The sifting property $\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a)$

Laplace transforms of impulse functions

$$\mathscr{L}\{\delta(t-a)\} = \mathrm{e}^{-as} \qquad \mathscr{L}^{-1}\{\mathrm{e}^{-as}\} = \delta(t-a)$$

$$\mathscr{L}^{-1}\{1\} = \delta(t) \qquad \mathscr{L}\{\delta(t)\} = 1$$

Relationship between Heaviside step and impulse functions

$$\delta(t-a) = \frac{\mathrm{d}}{\mathrm{d}t}H(t-a) = H'(t-a)$$

$$\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}H(t) = H'(t) \qquad f'(t) = g'(t) + \sum_{i=1}^{n} d_i\delta(t-t_i)$$

$$\int_{-\infty}^{\infty} f(t)\delta'(t-a)\,\mathrm{d}t = -f'(a)$$

The initial-value theorem $\lim_{t\to 0^+} f(t) = f(0^+) = \lim_{s\to\infty} sF(s)$

The final-value theorem $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

The Heaviside step function

$$f(t) = \begin{cases} f_1(t) & (0 \leqslant t < t_1) \\ f_2(t) & (t_1 \leqslant t < t_2) \\ f_3(t) & (t \geqslant t_2) \end{cases}$$

$$f(t) = f_1(t)H(t) + [f_2(t) - f_1(t)]H(t - t_1) + [f_3(t) - f_2(t)]H(t - t_2)$$

Laplace Transform for SISO systems

$$\dot{x} = Ax + bu \qquad y = c^T x$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} \quad \frac{\leftarrow c}{\leftarrow \boldsymbol{A}}$$

$$X(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}x(0) + (s\boldsymbol{I} - \boldsymbol{A})^{-1}bU(s)$$

$$x(t) = \mathcal{L}^{-1}\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}\}x(0) + \mathcal{L}^{-1}\{(s\boldsymbol{I} - \boldsymbol{A})^{-1}bU(s)\}$$

$$Y(s) = c^T X(s)$$

Assuming zero initial conditions

$$X(s) = (s\boldsymbol{I} - \boldsymbol{A})^{-1}bU(s) \qquad Y(s) = c^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}bU(s)$$

$$G(s) = c^T(s\boldsymbol{I} - \boldsymbol{A})^{-1}b = \frac{c^T \operatorname{adj}(s\boldsymbol{I} - \boldsymbol{A})b}{\det(s\boldsymbol{I} - \boldsymbol{A})}$$

Discriminant: $ax^2 + bx + c = 0$

$$x_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

quadratic form : multiplication of complex poles form

$$s^2 + 2.\sigma.s + (\sigma^2 + \omega^2) = [s + (\sigma + jw)].[s + (\sigma - jw)]$$

$$e^{j\theta} = \cos\theta + j\sin\theta,$$